Four-armed spiral tiling of scalene triangles

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Introduction

Robert Fathauer presented in [1] a tessellation based on a four-armed spiral. The triangles in that tiling have a right angle. In this paper a generalization of the tessellation is addressed using scalene triangles, that can be acute and obtuse. We show that for any given set of triangle angles there is only a single space filling configuration.

Analysis

Figure 1 sketches the addressed spiral configuration based on an arbitrary triangle. We will show that the two drawn iterations can be extended by smaller counter-clockwise iterations and bigger clockwise iterations, filling the whole plane.

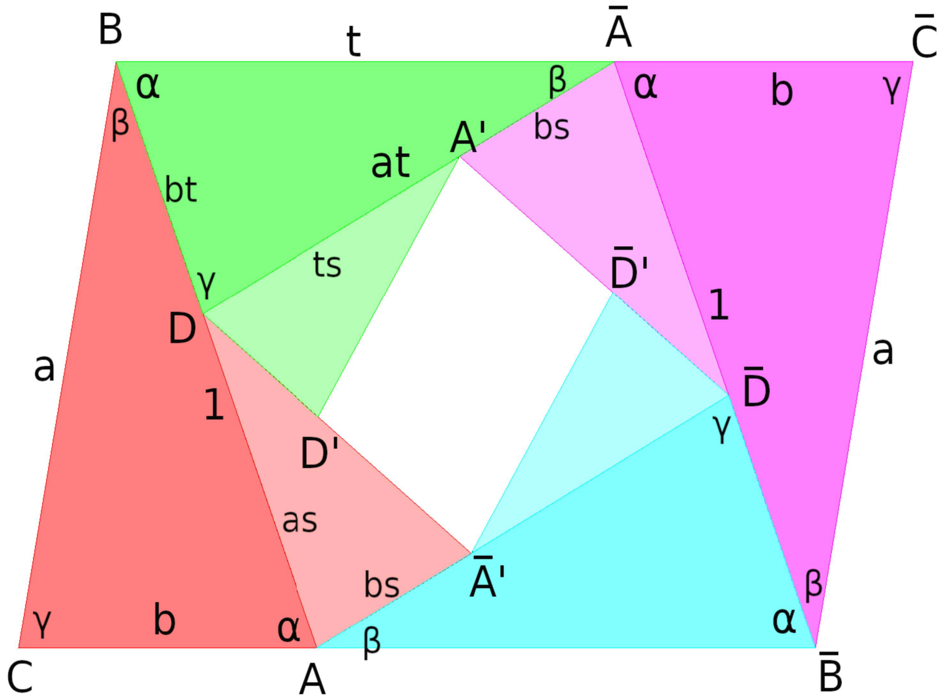

Figure 1: Spiral configuration

The 4 seed triangles at the border of the parallelogram are ABC (red), $\overline{ABC}$ (purple), $B\bar{A}D$ (green), and $\bar{B}A\bar{D}$ (blue). These 4 triangles have angles called : α , β , γ . Note that for triangles ABC and $\overline{ABC}$ the order of these 3 angles is counter-clockwise, whereas for triangles $B\bar{A}D$ and $\bar{B}A\bar{D}$ the order is clockwise: the latter 2 triangles are flipped compared to the former 2 triangles. Triangles ABC and $\overline{ABC}$ have the same size with sides equal to *a*, *b* and 1. Due to the normalization of side AB to 1 the values of *a* and *b* can be computed from the angles. The size of triangles $B\bar{A}D$ and $\bar{B}A\bar{D}$ differs a factor *t* with triangle ABC, so that their sides equal *a***t*, *b***t* and *t*. The middle of the parallelogram, considered as origin, is a point of symmetry. Triangles ABC and $\overline{ABC}$ are rotation symmetric with respect to the origin, and the same holds for triangles $B\bar{A}D$ and $\bar{B}A\bar{D}$.

Consider parallelogram AD$\overline{\mathrm{AD}}$, that has the same angles as parallelogram CB$\overline{\mathrm{CB}}$ at their corresponding corners. This can be easily verified using the property: $\alpha+\beta+\gamma=\pi$. Furthermore, we demand that the 2 parallelograms are similar. The scale factor between both parallelograms equals *s*. So side AD is *s* times BC, and side $\overline{\mathrm{A}}$D is *s* times $\overline{\mathrm{CB}}$. The latter gives the equation:

(1) $$a*t=s*(t+b)$$

Also side AB imposes a restriction to *s* and *t* :

(2) $$1=a*s+b*t$$

Combining equations (1) and (2) gives a quadratic equation in s (or t) :

(3) $$a*s^2-(a^2+b^2+1)*s+a=0$$

The discriminant of (3) is always positive, so that there are 2 real, different roots. The product of the roots is 1. So, one root is bigger than 1 and the other, desired one is smaller than 1. This proves that for any given set of triangle angles there is a unique pair of parameters *s* and *t*.

For example, the configuration in [1] :

(4) $$\alpha=\pi/6,\beta=\pi/3,\gamma=\pi/2,a=½*\sqrt{3},b=½$$

yields

(5) $$s=\frac{1}{\sqrt{3}},t=1$$

Parallelogram AD$\overline{\mathrm{AD}}$ is obtained from parallelogram CB$\overline{\mathrm{CB}}$ by scaling with a factor *s* and rotating by angle β . This process can be repeated unlimited. Parallelogram CB$\overline{\mathrm{CB}}$ can in the same way repeatedly be enlarged by scaling with a factor 1/*s* and rotating by angle $-\beta$, filling the whole plane.

References

[1] Robert W. Fathauer, “New tessellation based on a four-armed spiral tiling of right triangles.”, https://twitter.com/RobFathauerArt/status/1358449514244280320