Logarithmic spiral tiling of hexagons

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Introduction

Robert Fathauer presents in [1] a logarithmic spiral tiling of quadrilaterals. In this paper an analoguous kind of tessellation is addressed using hexagons having all similar shape. We also give examples of degenerated hexagons yielding triangles and quadrilaterals.

Analysis

Figure 1 shows a logarithmic spiral configuration of hexagons. In this example the number of spiral arms are given by N=7 and M=3.

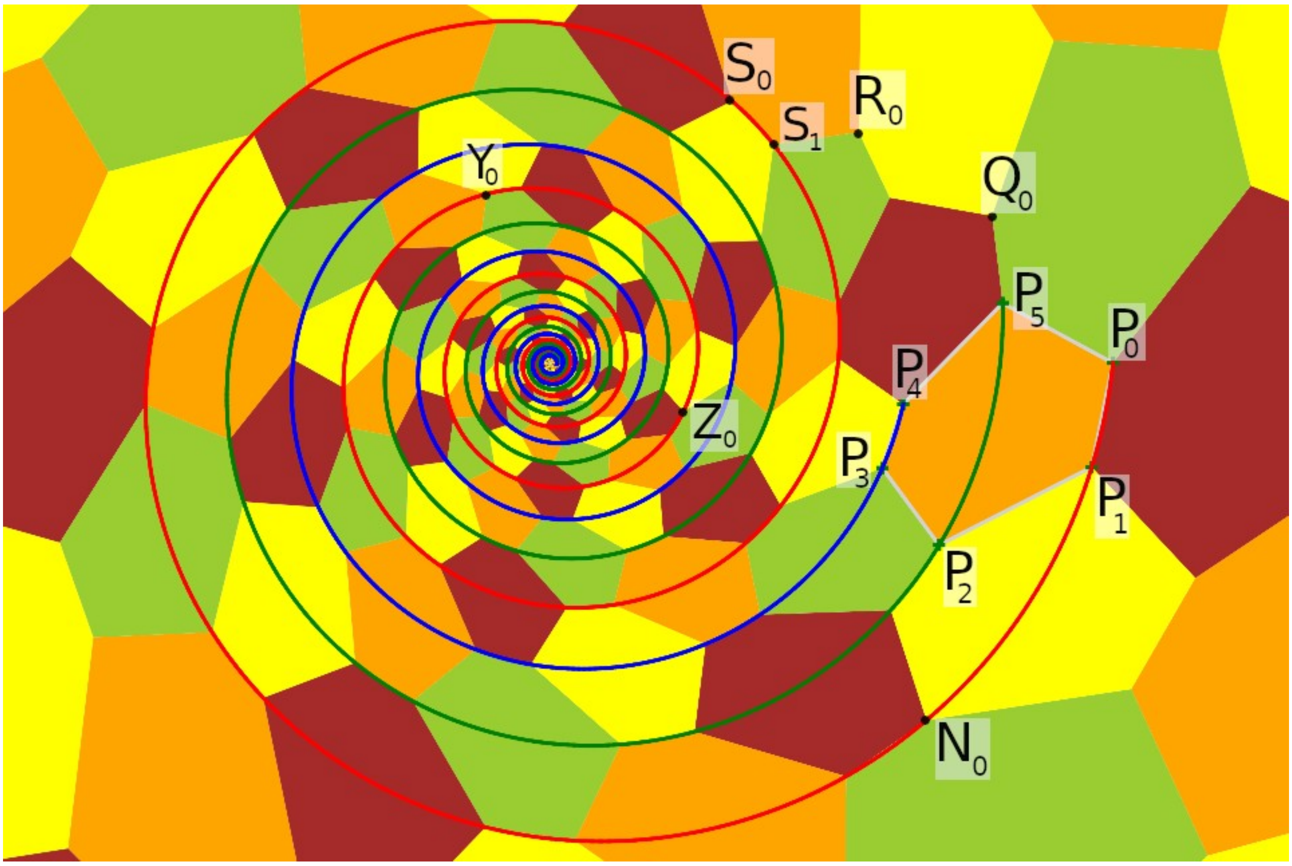

Figure 1: Spiral configuration

The vertices of all hexagons lie on the M spirals. In case of a single spiral (M=1) it is obvious that all vertices must lie on this spiral. In case of M=2 the 2 pairs of vertices that lie the fartest away from the spiral center and the closest to the center are on the same spiral, whereas the 2 middle vertices lie on the other spiral. For 3 or more spirals the vertices of a hexagon lie on 3 different spirals.

In the figure the "base" hexagon has a grey border. The spirals decrease in clockwise direction to the center. From the figure can be observed that there are N rays of hexagons towards the center, with N-1 rays in between those rays, and M-1 rays between the base hexagon and the N^{th} hexagon after the first revolution indicated by S_0.

The angle α is the rotation angle around the center rotating along the top edge of the hexagon from P_0 to P_1.(red spiral). The rotation angle around the center rotating along the bottom edge from P_4 to P_3 (blue spiral) has the same size α. The angle β is the rotation angle around the center rotating along at the middle of the hexagon from P_5 to P_2 (green spiral). If a hexagon rotates around the center along the 3 colored spirals over the sum angle γ it arrives at the next hexagon. Hence, the sum angle is written as:

(1) $$\gamma=\alpha+\beta$$

Rotating P_0 over the angle $N*\gamma$ brings us to S_0. If there were only 1 spiral (M=1) then S_1 would coincide with P_5: the hexagons would lie against each other. With M bigger than 1, there are M-1 hexagons in between.
The computation of the size of γ is easier when making another revolution, in this example from S_0 to Y_0. After the 2 revolutions there are still M rotations over angle γ needed for having the same direction to the center, in this case from Y_0 to Z_0. However, for an extra swing in the rays, we introduce an artificial extra skew parameter. So, γ satisfies:

(2) $$(N+\frac{M}{2})*\gamma=2*\pi+skew$$

There are no restrictions for α and β, apart from the fact that they must be non-negative.
The exponential growth parameter *k* will be derived from the ratio ρ between horizontal and vertical characteristics of the hexagon.

(3) $$\rho=\frac{|P_0-P_4|}{|P_0|*\gamma}$$

In concreto, this ratio is the quotient of the distance between points P_0 and P_4, and on the other hand the circular curve along P_0 over an angle of γ; the latter curve approximates the spiral curve from P_0 to N_0. This choice of the ratio has a clear human interpretation.
The points P_0, Q_0, R_0, and S_0 lie on a logarithmic spiral by construction. The factor F between subsequent points is based on the quotient between P_0 and S_0:

(4) $$F^M=\frac{S_0}{P_0}$$

S_0 has been obtained from P_0 by rotating over $N*\gamma$. Hence:

(5) $$S_0=P_0*e^{(k+i)*N*\gamma}$$

Bear in mind that the 4 mentioned points traverse in opposite direction compared to the spiral path from P_0 to S_0. For this reason the rotation angle is computed as a negative angle by subtracting an extra $2*\pi$. Taking this into account and combining (4) and (5) gives:

(6) $$F^M=e^{k*N*\gamma}*e^{i*(N*\gamma-2*\pi)}$$

or:

(7) $$F=e^{k*N*\gamma/M}*e^{i*(N*\gamma-2*\pi)/M}$$

Using *F*, formulas for all vertices of the base hexagon can be derived:

(8a) $$P_1=P_0*e^{(k+i)*\alpha}$$

(8b) $P_5 = Q_0 * e^{(k+i)*\alpha} = P_0 * F * e^{(k+i)*\alpha}$

(8c) $P_2 = P_5 * e^{(k+i)*\beta}$

(8d) $P_4 = R_0 * e^{(k+i)*\gamma} = P_0 * F^2 * e^{(k+i)*\gamma}$

(8e) $P_3 = P_4 * e^{(k+i)*\alpha}$

Substitution of (8d) in (3) leads to an expression for *k* :

(9) $$\rho = \frac{\left|1 - e^{k*\gamma*(2*N/M+1)} * e^{i*skew*2/M}\right|}{\gamma}$$

For convenience, define the following variables:

(10) $\lambda = e^{k*\gamma*(2*N/M+1)}$

(11) $D = skew*2/M$

With these we can convert (9) to:

(12) $(\rho*\gamma)^2 = 1 - 2*\lambda*\cos(D) + \lambda^2$

For solving this quadratic equation we take the root with the negative sign, because that agrees with the simple case of zero skew:

(13) $\lambda_1 = \cos(D) - \sqrt{\cos(D)^2 - (1 - (\rho*\gamma)^2)}$

So, the growth factor *k* becomes:

(14) $$k = \frac{\log(\lambda_1)}{\gamma*(2*N/M+1)}$$

Because the value of λ_1 is smaller than 1, the “growth” factor *k* has a negative value. This means that the base hexagon becomes smaller and smaller when rotating towards the center, as expected.

Special case: quadrilateral

When α is chosen to be zero, the points P_0 and P_1 overlap, and also the points P_3 and P_4 overlap. As a consequence the hexagon becomes a quadrilateral. Figure 2 shows two examples of degenerated hexagons becoming a quadrilateral; in Figure 2a the spiral has the same number of spiral arms as above: N=7 and M=3, whereas in Figure 2b the spiral has N=3 and M=6 resembling the structure of Fathauer's ducks spiral [2].

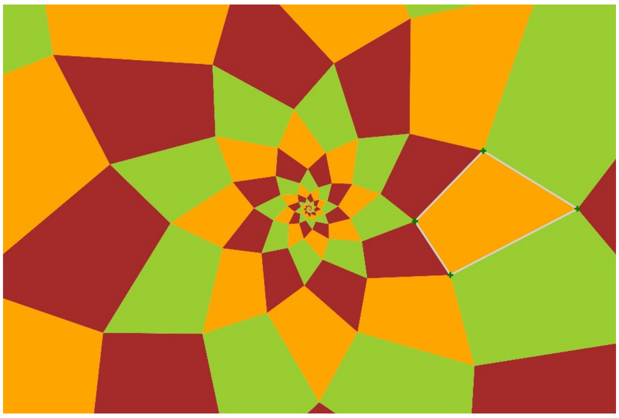

Figure 2a: Quadrilateral spiral: N=7, M=3

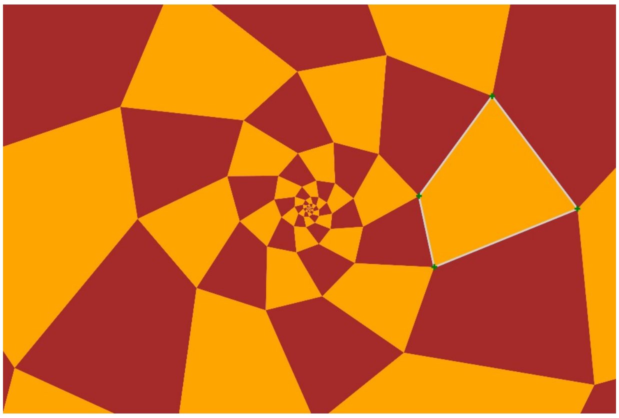

Figure 2b: Quadrilateral spiral: N=3, M=6

Special case: triangle

If angle β equals zero, the points P_2 and P_5 overlap. The hexagon becomes then a pair of triangles like a diabolo. Figure 3a shows the resulting spiral with the same number of spiral arms as above: N=7 and M=3. Figure 3b shows a version with deformed edges.

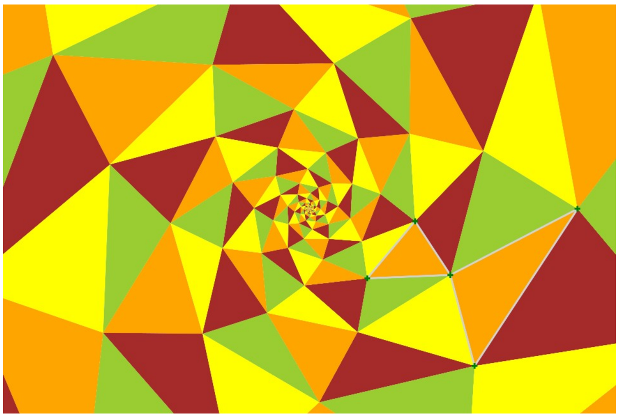

Figure 3a: Triangle spiral: N=7, M=3

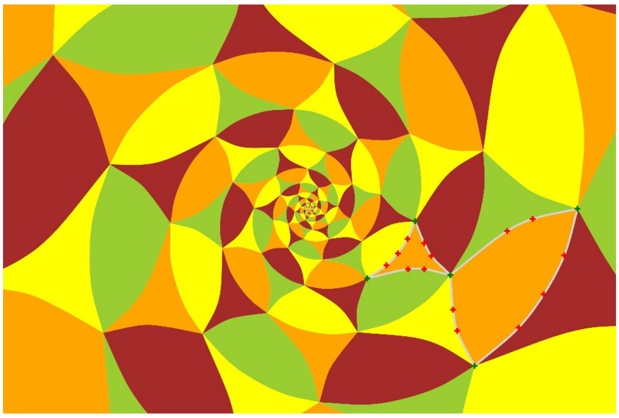

Figure 3b: Deformed triangle spiral: N=7, M=3

References

[1] Robert W. Fathauer, "Tessellations: Mathematics, Art, and Recreation", AK Peters/CRC Press, 2021

[2] Robert W. Fathauer, "New duck tessellation based on a spiral tiling of quadrilaterals", https://twitter.com/robfathauerart/status/1355884521120382984, 31 Januari 2021