# The computation of the spiral center of triangles and quadrilaterals

Tis Veugen
tis.veugen@gmail.com
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## *Introduction*

Spiral tilings of triangles have been studied by Waldman [1] and Fathauer [2]. This short memo deals with the computation of the spiral center given a triangle suitable for tessellation. The benefit of knowing the spiral center in advance is simple computation of gnomon corners based on the spiral's scale factor and rotation angle. Also, the drawing can then be centered at any wanted position.
Furthermore, it turns out that the spiral center of quadrilaterals as designed by Fathauer [3] can be computed in the same way as for triangles.

## *Triangles*

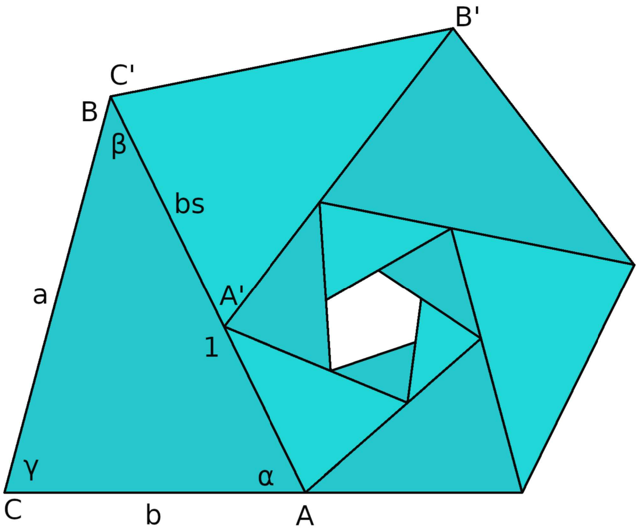


*Figure 1: Spiral with seed triangle*

The seed triangle has corners A, B, C; the angles at these corners are : $\alpha$ , $\beta$ , $\gamma$ . The opposite sides have length a, b, and 1.

Without loss of generality the side from A to C is horizontal. This eases to express the corners with complex variables. From the context should be clear if a capital is a corner or its corresponding complex number. Now, B and C can be expressed relative to A.

(1) $B = A + 1 * e^{i*(\pi - \alpha)}$
(2) $C = A - b$

The spiral is constructed by adding a smaller triangle A'B'C', being the first gnomon, next to ABC. The scale factor is called *s*; see [2] for its computation. The rotation angle $\alpha$ is in clockwise direction. The number of gnomons until alignment with the seed triangle equals *n* = 5 for the above figure. This number is not needed for the way of computation of the spiral center. Side A'C' is part of AB, such that corner C' is equal to B. The corners of A'B'C' can be expressed in terms of ABC.

(3) $A'=A+(1-b*s)*(B-A)$

(4) $B'=B+s*e^{-i*\alpha}*(B-C)$

(5) $C'=B$

The expression for B' characterizes mostly the spiral construction, since it includes the scale factor *s* and the (negative) rotation angle $\alpha$. Therefore, the characteristic spiral transformation factor is introduced:

(6) $\sigma=s*e^{-i*\alpha}$

The 3 corners are now combined in a tuple T of 3 elements

(7) $$T=\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

The transformed tuple T' can then be expressed in matrix notation:

(8) $T'=M*T$

with

(9) $$M=\begin{pmatrix} b*s & 1-b*s & 0 \\ 0 & 1+\sigma & -\sigma \\ 0 & 1 & 0 \end{pmatrix}$$

With this transformation matrix M any subsequent iteration of the seed tuple T can be easily computed, yielding the corners of the successive gnomons. Moreover, the center of the spiral can be derived by multiplying M infinite times. To perform this computation the eigendecomposition of M is calculated. The diagonal matrix L of eigenvalues for M is:

(10) $$L=\begin{pmatrix} b*s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

The corresponding matrix V of eigenvectors for M is:

(11) $$V=\begin{pmatrix} 1 & 1 & \sigma*(1-b*s) \\ 0 & 1 & \sigma*(\sigma-b*s) \\ 0 & 1 & \sigma-b*s \end{pmatrix}$$

So, M can now be written as:

(12) $M=V*L*V^{-1}$

where

(13) $$V^{-1}=\frac{1}{(1-\sigma)*(\sigma-b*s)}\begin{pmatrix} (1-\sigma)*(\sigma-b*s) & b*s*(1-\sigma) & -\sigma*(1-\sigma) \\ 0 & \sigma-b*s & -\sigma*(\sigma-b*s) \\ 0 & -1 & 1 \end{pmatrix}$$

The matrix $M^{\infty}$ after infinite multiplications becomes:

(14) $$M^{\infty}=(V*L*V^{-1})^{\infty}=V*L^{\infty}*V^{-1}=V*\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}*V^{-1}$$

or

$$(15) \qquad M^{\infty}=\frac{1}{(1-\sigma)}\begin{pmatrix}0 & 1 & -\sigma\\ 0 & 1 & -\sigma\\ 0 & 1 & -\sigma\end{pmatrix}$$

The 3 rows of $M^{\infty}$ are the same because the 3 corners converge to the same spiral center. For the computation of the center the tuple T is expressed relative to A:

$$(16) \qquad T=\begin{pmatrix}A\\ A\\ A\end{pmatrix}+\begin{pmatrix}0\\ -\sigma/s\\ -b\end{pmatrix}$$

Now the spiral center S can be computed using the first row $M_1^{\infty}$ of $M^{\infty}$ :

$$(17) \qquad S=M_1^{\infty}*T=A+\frac{-\sigma/s+\sigma*b}{1-\sigma}=A+\frac{1-b*s}{s-e^{i*\alpha}}$$

So, spiral center $\Delta$ relative to corner A equals:

$$(18) \qquad \Delta=\frac{1-b*s}{s-e^{i*\alpha}}$$

If the spiral center should lie at the origin, then take A such that $A=-\Delta$ . Changing A is in fact a translation of the whole spiral figure. The advantage of having the spiral center at the origin is that the next iterated corner can be easily computed by multiplication with the characteristic spiral transformation factor $\sigma$ . This can also be verified by comparing the elements of $T_s$ defined as

$$(19) \qquad T_s=\begin{pmatrix}-\Delta\\ -\Delta\\ -\Delta\end{pmatrix}+\begin{pmatrix}0\\ -\sigma/s\\ -b\end{pmatrix}$$

with the corresponding 3 elements of $M*T_s$ ; or in other words:

$$(20) \qquad M*T_s=\sigma*T_s$$

Hence, with induction holds for iteration *k* of the 3 corners:

$$(21) \qquad M^k*T_s=\sigma^k*T_s$$

Remark 1: Iterating in the “other” direction with growing triangles can be realized by using the inverse of M, or the reciprocal of $\sigma$ .

Remark 2: A dual tessellation can be formed based on the centers of the triangles. Also, any convex combination $f$, being a row vector of size 3, of the 3 corners can be used to generate a tessellation. Formula (21) can be used to show that the k-iterated convex point equals the initial convex point that has been transformed k times with $\sigma$ . Since:

$$(22) \qquad f*M^k*T_s=f*\sigma^k*T_s=\sigma^k*(f*T_s)$$

Remark 3: Formula (18) can be written in an alternative way to focus on the geometry for computing the spiral center.

$$(23) \qquad \Delta=\frac{1-b*s}{s-e^{i*\alpha}}=\frac{(1-b*s)*e^{-i*\alpha}}{s*e^{-i*\alpha}-1}=\frac{(1-b*s)*e^{i*(\pi-\alpha)}}{1-\sigma}$$

The numerator in (23) equals piece A’A, or in fact complex variable (A’ – A). The denominator is part of a geometric series with $\sigma$ as ratio. So, the spiral center can be constructed by repeatedly

scaling and rotating A'A and adding it to the partial sum.
As Fathauer [2] mentions, Figure 1 can be interpreted as n (=5) counter-clockwise spiral arms. One spiral arm is highlighted in red in Figure 2.

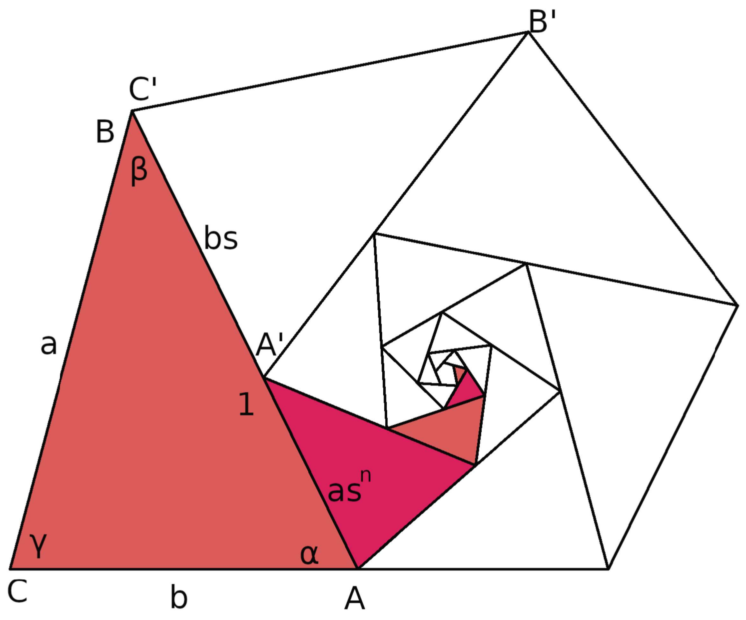


*Figure 2: Single spiral arm*

The scale factor for the successive triangles is now $s^n$ and the rotation angle is $\beta$ in counter-clockwise direction. Focussing on side AC, this piece is repeatedly scaled and rotated to construct a spiral. So relatively to C the spiral parts are:

(24) $$b+b*\tau+b*\tau^2+b*\tau^3+...=\frac{b}{1-\tau}$$

where:

(25) $$\tau=s^n*e^{i*\beta}$$

The angles $\alpha$ and $\beta$ are related as $n*\alpha+\beta=2*\pi$ , see also [2]. So

(26) $$\tau=s^n*e^{i*(2*\pi-n*\alpha)}=s^n*e^{i*(-n*\alpha)}=\sigma^n$$

Relatively to A the spiral center for the arm is:

(27) $$\Delta_1=\frac{b*\tau}{1-\tau}$$

From the figure is clear that $\Delta=\Delta_1$ . We can also show this in an algebraic way using the properties of the seed triangle (e.g. considering B as origin and rotating BC horizontally):

(28) $$a+b*e^{i*(\pi-\gamma)}=e^{i*\beta}$$ .

Now:

(29) $$\Delta_1=\frac{b*(a*s^n)*e^{i*\beta}}{a-(a*s^n)*e^{i*\beta}}=\frac{b*(1-b*s)*e^{i*\beta}}{a-(1-b*s)*e^{i*\beta}}=\frac{b*(1-b*s)*e^{i*\beta}}{b*e^{-i*\gamma}+(b*s)*e^{i*\beta}}=\frac{(1-b*s)}{e^{-i*(\gamma+\beta)}+s}$$

So

(30) $$\Delta_1=\frac{(1-b*s)}{e^{-i*(\gamma+\beta)}+s}=\frac{(1-b*s)}{e^{-i*(\pi-\alpha)}+s}=\frac{(1-b*s)}{s-e^{i*\alpha}}=\Delta$$

## *Quadrilaterals*

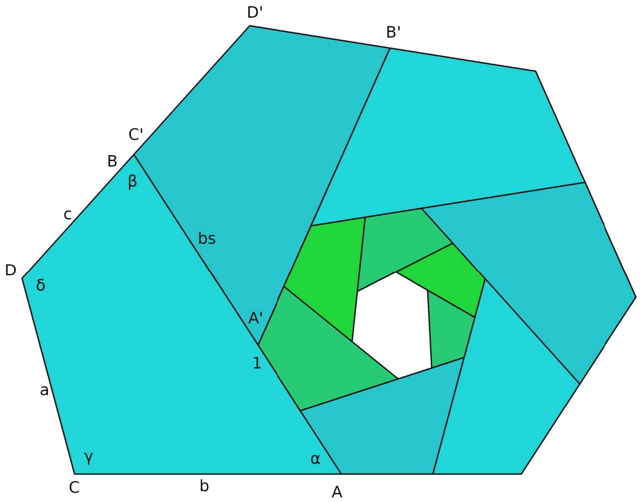


*Figure 3: Spiral with seed quadrilateral*

The nomenclature for the seed quadrilateral is analoguous to the seed triangle in the above Triangles section. For constructing the spiral the triangle ABC within the quadrilateral is iterated in exactly the same way as triangle ABC in the Triangles section, using the same variable names *b*, *s*, $\alpha$, $\sigma$, etc. For this reason the same formulas in the Triangles section can be applied for computing the spiral center of the iterated quadrilateral.

## *References*


[1] Cye H. Waldman, "Gnomon is an Island Whorled Polygons", http://old.nationalcurvebank.org/gnomon/Gnomon%20is%20an%20Island.pdf, 3 July 2016

[2] Robert W. Fathauer, "Logarithmic Spiral Tilings of Triangles", https://www.mathartfun.com/FathauerBridges2021v1.pdf, Submitted to the 2021 Bridges Conference

[3] Robert W. Fathauer, "Spiral tilings of quadrilaterals", https://twitter.com/RobFathauerArt/status/1360248744021618692, 12 Februari 2021